

Kondo effect in Dirac systems

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We investigate the Kondo effect in Dirac systems, where Dirac electrons interact with the localized spin via the s-d exchange coupling. The Dirac electron in solid state has the linear dispersion and is described typically by the Hamiltonian such as $H_k = v\mathbf{k} \cdot \boldsymbol{\sigma}$ for the wave number \mathbf{k} where σ_j are Pauli matrices. We derived the formula of the Kondo temperature T_K by means of the Green's function theory for small J . The T_K is determined from a singularity of Green's functions in the form $T_K \simeq \bar{D} \exp(-\text{const.}/\rho|J|)$ when the exchange coupling $|J|$ is small where $\bar{D} = D/\sqrt{1 + D^2/(2\mu)^2}$ for a cutoff D and ρ is the density of states at the Fermi surface. When $|\mu| \ll D$, T_K is proportional to $|\mu|$: $T_K \simeq |\mu| \exp(-\text{const.}/\rho|J|)$. The Kondo screening will, however, disappear when the Fermi surface shrinks to a point called the Dirac point, that is, T_K vanishes when the chemical potential μ is just at the Dirac point. The resistivity and the specific heat exhibit a \log - T singularity in the range $T_K < T \ll |\mu|/k_B$. Instead, for $T \sim O(|\mu|)$ or $T > |\mu|$, they never show \log - T .

I. INTRODUCTION

Recently, the Dirac electron in solid state has been investigated intensively because the Dirac cones in metals or semimetals have been realized in solids[1–5]. In several materials such as Graphene[6–10] as well as Bismuth compounds, there appear the conduction bands with the linear dispersion that are described by Dirac Hamiltonian. The Dirac cone in the band structure appears in the surface states of topological insulators[11, 12]. Three-dimensional Dirac semimetals have also been realized in Na_3Bi [13] and Cd_3As_2 [14].

The Kondo effect, that occurs as the result of the exchange interaction between dilute magnetic impurities and conduction electrons, is one of the most important phenomena in solid state physics[15, 16]. It is interesting to examine how the Kondo effect emerges in Dirac systems. The linear dispersion of Dirac electrons would affect the characteristic features of the Kondo effect. It is not even trivial whether the Kondo effect indeed appears in Dirac systems. The purpose of this work is to show unique and interesting properties of the Kondo effect in these systems.

The s-d model in Dirac systems is closely related to the pseudogap model of the Kondo problem where magnetic impurities couple to conductive fermions with a density of states $\rho(\omega) \propto |\omega|^r$ ($0 < r$)[17–19]. In the pseudogap model, the density of states vanishes at the Fermi level. This feature is common to the Kondo problem in Dirac systems when the Fermi surface is point like, namely, the chemical potential is just at the Fermi point $\mu = 0$. It has been suggested that there is a phase transition between the local-moment phase and the strong-coupling singlet phase in the pseudogap Kondo and Anderson models[18]. A Dirac system is also interesting from the viewpoint of topology because the index theorem has been proved for Dirac operators[20, 21]

In this paper we investigate the s-d Hamiltonian in a Dirac system by means of the Green's function theory and evaluate the Kondo temperature T_K . We employ

the decoupling procedure to obtain a closed solution for a set of equations for Green's functions. Although the decoupling procedure is valid only for small J in the region, $T > T_K$, this method is useful to derive the Kondo temperature. It turns out that T_K is crucially dependent on the chemical potential μ . T_K vanishes for $\mu = 0$ at the Dirac point in our method. This shows the absence of Kondo screenings at $\mu = 0$ and suggests that there may be a transition at some critical value of $J = J_c$ from the local-moment phase to the singlet-formation phase at $\mu = 0$. In Dirac systems T_K is finite and is given by the formula being similar to that in the conventional Kondo effect: $T_K \propto \bar{D} \exp(-\text{const.}/\rho|J|)$ when $\rho|J|$ is small where ρ is the density of states and J is the exchange coupling constant. When $\rho|J|$ is large of order 1, T_K is given by an algebraic function of $\rho|J|$ like $T_K \propto |\mu|Q(\rho|J|)$ for a some function $Q(x)$.

The paper is organized as follows. In section II we show the Hamiltonian. In section III Green's functions are defined and a set of equations for them are derived. In section IV the Kondo temperature is derived. In sections V and VI, we evaluate the electric resistivity and specific heat, respectively, and discuss their \log - T singularity. The last section is devoted to a summary.

II. HAMILTONIAN

We consider a massless Dirac Hamiltonian (Weyl Hamiltonian):

$$H_D = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger (v_x k_x \sigma_x + v_y k_y \sigma_y + v_z k_z \sigma_z) \psi_{\mathbf{k}}, \quad (1)$$

where

$$\psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{\mathbf{k}\downarrow} \end{pmatrix} \quad (2)$$

with the annihilation and creation operators $c_{\mathbf{k}\sigma}$ and $c_{\mathbf{k}\sigma}^\dagger$, respectively. Here v_x , v_y and v_z are velocities of conduction electrons. σ_x , σ_y and σ_z are Pauli matrices. In our

model the Dirac electrons, described by this Hamiltonian, interact with the localized spin. The total Hamiltonian is written as $H = H_0 + H_{sd}$ where

$$H_0 = \sum_{\mathbf{k}} [(v_x k_x - i v_y k_y) c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\downarrow} + (v_x k_x + i v_y k_y) c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}\uparrow} + v_z k_z (c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow} - c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}\downarrow}) - \mu (c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow} + c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}\downarrow})], \quad (3)$$

$$H_{sd} = -\frac{J}{2} \frac{1}{N} \sum_{\mathbf{k}\mathbf{k}'} [S_z (c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}'\uparrow} - c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}'\downarrow}) + S_+ c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}'\uparrow} + S_- c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}'\downarrow}]. \quad (4)$$

N is the number of sites and we have included the chemical potential μ . S_+ , S_- and S_z denote the operators of the localized spin. The term H_{sd} indicates the s-d interaction between the conduction electrons and the localized spin, with the coupling constant J [15, 16]. J is negative, as adopted in this paper, for the antiferromagnetic interaction. The Dirac Hamiltonian resembles the s-d model with the spin-orbit coupling of Rashba type[22]. We use the Green's function method, following Ref.[22] to evaluate the Kondo temperature and its related properties.

III. GREEN'S FUNCTIONS

We define thermal Green's functions[23, 24]:

$$G_{\mathbf{k}\mathbf{k}'\sigma}(\tau) = -\langle T_\tau c_{\mathbf{k}\sigma}(\tau) c_{\mathbf{k}'\sigma}^\dagger(0) \rangle, \quad (5)$$

$$F_{\mathbf{k}\mathbf{k}'}(\tau) = -\langle T_\tau c_{\mathbf{k}\downarrow}(\tau) c_{\mathbf{k}'\uparrow}^\dagger(0) \rangle, \quad (6)$$

$$\langle \langle S_z c_{\mathbf{k}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau = -\langle T_\tau S_z c_{\mathbf{k}\uparrow}(\tau) c_{\mathbf{k}'\uparrow}^\dagger(0) \rangle, \quad (7)$$

$$\langle \langle S_- c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau = -\langle T_\tau S_- c_{\mathbf{k}\downarrow}(\tau) c_{\mathbf{k}'\uparrow}^\dagger(0) \rangle, \quad (8)$$

$$\langle \langle S_z c_{\mathbf{k}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau = -\langle T_\tau S_z c_{\mathbf{k}\uparrow}(\tau) c_{\mathbf{k}'\uparrow}^\dagger(0) \rangle, \quad (9)$$

$$\langle \langle S_- c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau = -\langle T_\tau S_- c_{\mathbf{k}\downarrow}(\tau) c_{\mathbf{k}'\uparrow}^\dagger(0) \rangle. \quad (10)$$

Here T_τ is the time ordering operator. The Fourier transforms are defined as usual:

$$G_{\mathbf{k}\mathbf{k}'\sigma}(\tau) = \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} G_{\mathbf{k}\mathbf{k}'\sigma}(i\omega_n), \quad (11)$$

$$F_{\mathbf{k}\mathbf{k}'}(\tau) = \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} F_{\mathbf{k}\mathbf{k}'}(i\omega_n), \quad (12)$$

$$\langle \langle S_z c_{\mathbf{k}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau = \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} \langle \langle S_z c_{\mathbf{k}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n}, \quad (13)$$

$$\langle \langle S_- c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau = \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} \langle \langle S_- c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n}. \quad (14)$$

We also use the following Green's function,

$$\begin{aligned} \Gamma_{\mathbf{k}\mathbf{k}'\uparrow}(\tau) &= \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} \Gamma_{\mathbf{k}\mathbf{k}'\uparrow}(i\omega_n) \\ &= \langle \langle S_z c_{\mathbf{k}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau + \langle \langle S_- c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau. \end{aligned} \quad (15)$$

We start from the commutation relations:

$$[H_0, c_{\mathbf{k}\uparrow}] = -(v_x k_x - i v_y k_y) c_{\mathbf{k}\downarrow} + (-v_z k_z + \mu) c_{\mathbf{k}\uparrow}, \quad (16)$$

$$[H_0, c_{\mathbf{k}\downarrow}] = -(v_x k_x + i v_y k_y) c_{\mathbf{k}\uparrow} + (v_z k_z + \mu) c_{\mathbf{k}\downarrow}, \quad (17)$$

$$[H_{sd}, c_{\mathbf{k}\uparrow}] = -\frac{J}{2N} \sum_{\mathbf{k}'} (-S_z c_{\mathbf{k}'\uparrow} - S_- c_{\mathbf{k}'\downarrow}), \quad (18)$$

$$[H_{sd}, c_{\mathbf{k}\downarrow}] = -\frac{J}{2N} \sum_{\mathbf{k}'} (S_z c_{\mathbf{k}'\downarrow} - S_+ c_{\mathbf{k}'\uparrow}). \quad (19)$$

The equations of motion for $G_{\mathbf{k}\mathbf{k}'\uparrow}(\tau)$ and $F_{\mathbf{k}\mathbf{k}'}$ are

$$\begin{aligned} \frac{\partial}{\partial \tau} G_{\mathbf{k}\mathbf{k}'\uparrow}(\tau) &= -\delta(\tau) \delta_{\mathbf{k}\mathbf{k}'} + (-v_z k_z + \mu) G_{\mathbf{k}\mathbf{k}'\uparrow}(\tau) \\ &\quad - (v_x k_x - i v_y k_y) F_{\mathbf{k}\mathbf{k}'}(\tau) \\ &\quad + \frac{J}{2N} \sum_{\mathbf{q}} [\langle \langle S_z c_{\mathbf{q}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau + \langle \langle S_- c_{\mathbf{q}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau], \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{\partial}{\partial \tau} F_{\mathbf{k}\mathbf{k}'}(\tau) &= (v_z k_z + \mu) F_{\mathbf{k}\mathbf{k}'} - (v_x k_x + i v_y k_y) G_{\mathbf{k}\mathbf{k}'\uparrow}(\tau) \\ &\quad - \frac{J}{2N} \sum_{\mathbf{q}} [\langle \langle S_z c_{\mathbf{q}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau - \langle \langle S_+ c_{\mathbf{q}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_\tau]. \end{aligned} \quad (21)$$

Then the equation for $G_{\mathbf{k}\mathbf{k}'\uparrow}$ reads

$$\begin{aligned} (i\omega_n + \mu - v_z k_z) G_{\mathbf{k}\mathbf{k}'\uparrow}(i\omega_n) &= \delta_{\mathbf{k}\mathbf{k}'} \\ &\quad + (v_x k_x - i v_y k_y) F_{\mathbf{k}\mathbf{k}'}(i\omega_n) - \frac{J}{2N} \sum_{\mathbf{q}} \Gamma_{\mathbf{q}\mathbf{k}'\uparrow}(i\omega_n). \end{aligned} \quad (22)$$

The equations for $\langle \langle S_z c_{\mathbf{k}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle$ and $\langle \langle S_- c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle$ are similarly obtained as

$$\begin{aligned} (i\omega_n - v_z k_z + \mu) \langle \langle S_z c_{\mathbf{k}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} &= \langle S_z \rangle \delta_{\mathbf{k}\mathbf{k}'} \\ &\quad + (v_x k_x - i v_y k_y) \langle \langle S_z c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} \\ &\quad - \frac{J}{2N} \sum_{\mathbf{q}} \left[\langle \langle S_z^2 c_{\mathbf{q}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} + \frac{1}{2} \langle \langle S_- c_{\mathbf{q}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} \right] \\ &\quad - \frac{J}{2N} \sum_{\mathbf{q}\mathbf{q}'} \left[\langle \langle S_+ c_{\mathbf{k}\uparrow} c_{\mathbf{q}\downarrow}^\dagger c_{\mathbf{q}'\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} \right. \\ &\quad \left. - \langle \langle S_- c_{\mathbf{k}\uparrow} c_{\mathbf{q}\uparrow}^\dagger c_{\mathbf{q}'\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} \right], \end{aligned} \quad (23)$$

$$\begin{aligned}
& (i\omega_n + v_z k_z + \mu) \langle \langle S_- c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} \\
&= (v_x k_x + i v_y k_y) \langle \langle S_- c_{\mathbf{k}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} \\
&- \frac{J}{2N} \sum_{\mathbf{q}} \left[\langle \langle S_+ S_- c_{\mathbf{q}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} + \frac{1}{2} \langle \langle S_- c_{\mathbf{q}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} \right] \\
&- \frac{J}{2N} \sum_{\mathbf{q}\mathbf{q}'} \left[\langle \langle S_- c_{\mathbf{k}\downarrow} c_{\mathbf{q}\uparrow}^\dagger c_{\mathbf{q}'\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} \right. \\
&- \left. \langle \langle S_- c_{\mathbf{k}\downarrow} c_{\mathbf{q}\downarrow}^\dagger c_{\mathbf{q}'\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} \right] \\
&+ \frac{J}{N} \sum_{\mathbf{q}\mathbf{q}'} \langle \langle S_z c_{\mathbf{k}\downarrow} c_{\mathbf{q}\downarrow}^\dagger c_{\mathbf{q}'\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n}. \quad (24)
\end{aligned}$$

In this paper, we adopt the decoupling procedure for Green's functions including several operators[22, 25–27]. For example,

$$\begin{aligned}
\langle \langle S_+ c_{\mathbf{k}\downarrow} c_{\mathbf{q}\downarrow}^\dagger c_{\mathbf{q}'\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle &\simeq \langle S_+ c_{\mathbf{q}\downarrow}^\dagger c_{\mathbf{q}'\uparrow} \rangle \langle \langle c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle \\
&- \langle S_+ c_{\mathbf{q}\downarrow}^\dagger c_{\mathbf{k}\uparrow} \rangle \langle \langle c_{\mathbf{q}'\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle. \quad (25)
\end{aligned}$$

We need further the Green's functions $\langle \langle S_z c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle$ and $\langle \langle S_- c_{\mathbf{k}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle$. We neglect the terms of the order of J in the equations of motion for these Green's functions; this means that we use the following approximation:

$$\langle \langle S_z c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} \simeq \frac{v_x k_x + i v_y k_y}{i\omega_n + v_z k_z + \mu} \langle \langle S_z c_{\mathbf{k}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n}, \quad (26)$$

$$\langle \langle S_- c_{\mathbf{k}\uparrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n} \simeq \frac{v_x k_x - i v_y k_y}{i\omega_n - v_z k_z + \mu} \langle \langle S_- c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n}. \quad (27)$$

For $F_{\mathbf{k}\mathbf{k}'} = \langle \langle c_{\mathbf{k}\downarrow}; c_{\mathbf{k}'\uparrow}^\dagger \rangle \rangle_{i\omega_n}$, we neglect the terms of the order of J in a similar way to obtain

$$F_{\mathbf{k}\mathbf{k}'}(i\omega_n) \simeq \frac{(v_x k_x + i v_y k_y)}{i\omega_n + v_z k_z + \mu} G_{\mathbf{k}\mathbf{k}'\uparrow}(i\omega_n) + O(J). \quad (28)$$

We set $\langle S_z \rangle = 0$. From eqs.(23) and (24), $\Gamma_{\mathbf{k}\mathbf{k}'\uparrow}$ is written as

$$\begin{aligned}
\Gamma_{\mathbf{k}\mathbf{k}'\uparrow}(i\omega_n) &= \frac{i\omega_n + \mu}{(i\omega_n + \mu)^2 - (v_x^2 k_x^2 + v_y^2 k_y^2 + v_z^2 k_z^2)} \\
&\times \left[- \left(\frac{3}{4} - m_{\mathbf{k}} \right) \frac{J}{2N} G_{\mathbf{k}'\uparrow}^0(i\omega_n) \right. \\
&+ \left(\frac{3}{4} - m_{\mathbf{k}} \right) \frac{J}{2N} \sum_{\mathbf{q}} G_{\mathbf{q}}^0(i\omega_n) \frac{J}{2N} \sum_{\mathbf{p}} \Gamma_{\mathbf{p}\mathbf{k}'\uparrow}(i\omega_n) \\
&- \left. \left(n_{\mathbf{k}\uparrow} + n_{\mathbf{k}\downarrow} - 1 \right) \frac{J}{2N} \sum_{\mathbf{p}} \Gamma_{\mathbf{p}\mathbf{k}'\uparrow}(i\omega_n) \right] + \dots, \quad (29)
\end{aligned}$$

where \dots indicates the terms, being proportional to k_z , which give small contributions to $G_{\mathbf{k}\mathbf{k}'\uparrow}$ because they would vanish when we take the summation with respect

to \mathbf{k} . Here we set

$$m_{\mathbf{k}} = 3 \sum_{\mathbf{q}} \langle c_{\mathbf{q}\uparrow}^\dagger c_{\mathbf{k}\downarrow} S_- \rangle, \quad (30)$$

$$n_{\mathbf{k}\sigma} = \sum_{\mathbf{q}} \langle c_{\mathbf{q}\sigma}^\dagger c_{\mathbf{k}\sigma} \rangle, \quad (31)$$

and define

$$G_{\mathbf{k}}^0(i\omega_n) = \frac{i\omega_n + \mu}{(i\omega_n + \mu)^2 - (v_x^2 k_x^2 + v_y^2 k_y^2 + v_z^2 k_z^2)}, \quad (32)$$

$$G_{\mathbf{k}\sigma}^0(i\omega_n) = \frac{i\omega_n + \mu + \sigma v_z k_z}{(i\omega_n + \mu)^2 - (v_x^2 k_x^2 + v_y^2 k_y^2 + v_z^2 k_z^2)}. \quad (33)$$

We also define the following functions:

$$F(i\omega_n) = \frac{1}{N} \sum_{\mathbf{k}} G_{\mathbf{k}}^0(i\omega_n), \quad (35)$$

$$\Gamma(i\omega_n) = \frac{1}{N} \sum_{\mathbf{k}} \left(m_{\mathbf{k}} - \frac{3}{4} \right) G_{\mathbf{k}}^0(i\omega_n), \quad (36)$$

$$G(i\omega_n) = \frac{1}{2N} \sum_{\mathbf{k}} (n_{\mathbf{k}\uparrow} + n_{\mathbf{k}\downarrow} - 1) G_{\mathbf{k}}^0(i\omega_n). \quad (37)$$

Then we obtain

$$\sum_{\mathbf{k}} \Gamma_{\mathbf{k}\mathbf{k}'\uparrow} = \frac{J}{2} \frac{\Gamma(i\omega_n) G_{\mathbf{k}'\uparrow}^0(i\omega_n)}{1 + JG(i\omega_n) + \left(\frac{J}{2} \right)^2 \Gamma(i\omega_n) F(i\omega_n)}. \quad (38)$$

Now we can obtain a closed solution for a set of Green's functions. The Green's function $G_{\mathbf{k}\mathbf{k}'\uparrow}(i\omega_n)$ reads

$$\begin{aligned}
G_{\mathbf{k}\mathbf{k}'\uparrow}(i\omega_n) &= \delta_{\mathbf{k}\mathbf{k}'} G_{\mathbf{k}\uparrow}^0(i\omega_n) - \frac{J}{2N} G_{\mathbf{k}\uparrow}^0(i\omega_n) \frac{J}{2} \Gamma(i\omega_n) \\
&\times G_{\mathbf{k}'\uparrow}^0(i\omega_n) \frac{1}{1 + JG(i\omega_n) + \left(\frac{J}{2} \right)^2 \Gamma(i\omega_n) F(i\omega_n)}. \quad (39)
\end{aligned}$$

The Green's function $G_{\mathbf{k}\mathbf{k}'\downarrow}(i\omega_n)$ is similarly obtained as

$$\begin{aligned}
G_{\mathbf{k}\mathbf{k}'\downarrow}(i\omega_n) &= \delta_{\mathbf{k}\mathbf{k}'} G_{\mathbf{k}\downarrow}^0(i\omega_n) - \frac{J}{2N} G_{\mathbf{k}\downarrow}^0(i\omega_n) \frac{J}{2} \Gamma(i\omega_n) \\
&\times G_{\mathbf{k}'\downarrow}^0(i\omega_n) \frac{1}{1 + JG(i\omega_n) + \left(\frac{J}{2} \right)^2 \Gamma(i\omega_n) F(i\omega_n)}. \quad (40)
\end{aligned}$$

IV. KONDO TEMPERATURE

From the Green's function $G_{\mathbf{k}\mathbf{k}'\uparrow}(i\omega_n)$, the Kondo temperature T_K is determined from a zero of the denominator in this formula. We perform the analytic continuation $i\omega_n \rightarrow \omega$ and consider

$$1 + JG(\omega) = 0 \quad (41)$$

in the limit $\omega \rightarrow 0$ by neglecting higher-order term being proportional to $(J/2)^2$. We neglect the term of the order

of J in $n_{\mathbf{k}\sigma} = \langle c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} \rangle$. The equation in eq.(41) is written as

$$1 = \frac{1}{4} J \frac{1}{N} \sum_{\mathbf{k}} G_{\mathbf{k}}^0(\omega) \left[\tanh\left(\frac{\epsilon_{\mathbf{k}} - \mu}{2k_{\text{B}}T_{\text{K}}}\right) + \tanh\left(\frac{-\epsilon_{\mathbf{k}} - \mu}{2k_{\text{B}}T_{\text{K}}}\right) \right], \quad (42)$$

where $\epsilon_{\mathbf{k}} = \sqrt{v_x^2 k_x^2 + v_y^2 k_y^2 + v_z^2 k_z^2}$. Let us adopt for simplicity that $v_x = v_y = v_z = v$ and the dispersion is $\xi_{\mathbf{k}} = \pm v \sqrt{k_x^2 + k_y^2 + k_z^2} - \mu$. We can consider, in general, the d -dimensional case; by setting $v_z = 0$ for two dimensions and $v_y = v_z = 0$ for one dimension. Then the equation for T_{K} is

$$1 = \frac{1}{8} J \frac{\Omega_d}{(2\pi)^d} \int_0^D dk k^{d-1} \left(\frac{1}{\omega + \mu - vk} + \frac{1}{\omega + \mu + vk} \right) \times T_{\text{K}} \sum_{n=-\infty}^{\infty} \left[\frac{1}{vk - \mu - i\pi(2n+1)T_{\text{K}}} - \frac{1}{vk + \mu + i\pi(2n+1)T_{\text{K}}} \right], \quad (43)$$

where Ω_d is the solid angle in d -dimensional space, namely, the area of the $(d-1)$ -sphere S^{d-1} . D is an cutoff and we set $k_{\text{B}} = 1$ for simplicity. When d is odd, the integrand is an even function of k . In this case the integral with respect to k is reduced to an integral in the complex plane. For $d = 3$, the equation in eq.(43) is written as

$$1 = \frac{1}{32\pi^2} J \left[\sum_{n \geq 0} \frac{1}{v} \left(\frac{\mu + i\pi(2n+1)T_{\text{K}}}{v} \right)^2 \times 2\pi i T_{\text{K}} \left(\frac{1}{\omega - i\pi(2n+1)T_{\text{K}}} + \frac{1}{\omega + 2\mu + i\pi(2n+1)T_{\text{K}}} \right) - \sum_{n < 0} \frac{1}{v} \left(\frac{\mu + i\pi(2n+1)T_{\text{K}}}{v} \right)^2 \times 2\pi i T_{\text{K}} \left(\frac{1}{\omega - i\pi(2n+1)T_{\text{K}}} + \frac{1}{\omega + 2\mu + i\pi(2n+1)T_{\text{K}}} \right) \right], \quad (44)$$

where the summation has the upper limit $n_0 \equiv D/(2\pi T_{\text{K}})$. By using the formula of the digamma function,

$$\sum_{n=0}^{n_0} \frac{1}{n + \frac{1}{2} + x} = \psi\left(\frac{1}{2} + x + n_0\right) - \psi\left(\frac{1}{2} + x\right), \quad (45)$$

we obtain

$$1 = \frac{1}{32\pi^2} |J| \frac{\mu^2}{v^3} \left[\psi\left(\frac{1}{2} - \frac{\omega}{2\pi i T_{\text{K}}} + n_0\right) - \psi\left(\frac{1}{2} - \frac{\omega}{2\pi i T_{\text{K}}}\right) - \psi\left(\frac{1}{2} + \frac{\omega + 2\mu}{2\pi i T_{\text{K}}} + n_0\right) + \psi\left(\frac{1}{2} + \frac{\omega + 2\mu}{2\pi i T_{\text{K}}}\right) + \psi\left(\frac{1}{2} + \frac{\omega}{2\pi i T_{\text{K}}} + n_0\right) - \psi\left(\frac{1}{2} + \frac{\omega}{2\pi i T_{\text{K}}}\right) - \psi\left(\frac{1}{2} - \frac{\omega + 2\mu}{2\pi i T_{\text{K}}} + n_0\right) + \psi\left(\frac{1}{2} - \frac{\omega + 2\mu}{2\pi i T_{\text{K}}}\right) \right]. \quad (46)$$

We assume that the cutoff D is much larger than the temperature: $D \gg T$. We employ the asymptotic form of $\psi(z) \sim \log(z)$ for large z . Then the T_{K} is the solution of the equation

$$1 = \frac{1}{16} \rho_{\text{d}}(\mu) |J| \left[K(t) + \log\left(\frac{D}{\sqrt{D^2 + 4\mu^2}}\right) \right], \quad (47)$$

where we defined $t \equiv T/|\mu|$ and

$$K(t) = \text{Re} \psi\left(\frac{1}{2} + i \frac{1}{\pi t}\right) - \psi\left(\frac{1}{2}\right). \quad (48)$$

We introduced the density of states ρ_{d} as

$$\rho_{\text{d}}(\mu) = \frac{\Omega_d}{(2\pi)^d} \left(\frac{|\mu|}{v} \right)^{d-1} \frac{1}{v}. \quad (49)$$

Here, Ω_d is the solid angle in d -dimensional space, that is, the area of the $(d-1)$ -sphere S^{d-1} . As t approaches 0, $K(t)$ behaves as $K(t) \sim \log(1/\pi t)$. For large t , $K(t) \sim 7\zeta(3)/(\pi^2 t^2)$ where $\zeta(3)$ is the Riemann zeta function at argument 3. The equation eq.(47) always has a solution for $|J| > 0$. When $\rho_{\text{d}}|J|$ is small, $\rho_{\text{d}}|J| \ll 1$, we have a solution in the logarithmic region of $K(t)$. Then we obtain the Kondo temperature,

$$k_{\text{B}}T_{\text{K}} = \frac{2e^{\gamma}D}{\pi} \frac{1}{\sqrt{1 + \frac{D^2}{4\mu^2}}} \exp\left(-\frac{8}{\rho_{\text{d}}(\mu)|J|}\right), \quad (50)$$

where γ is Euler's constant and k_{B} is included in the formula of T_{K} . The result shows that the Kondo effect indeed occurs in a Dirac system.

When $D \gg |\mu|$, T_{K} is

$$k_{\text{B}}T_{\text{K}} = \frac{4e^{\gamma}|\mu|}{\pi} \exp\left(-\frac{8}{\rho_{\text{d}}(\mu)|J|}\right). \quad (51)$$

In the limit $|\mu| \rightarrow 0$, the equation $1 + JG(0) = 0$ has no solution. This indicates that when $|\mu|$ is small, T_{K} is reduced and vanishes for $|\mu| \rightarrow 0$. Hence, when the Fermi surface is point like, the Kondo effect never appears; this is because the scattering from the localized spin becomes weak for the point-like Fermi surface. This is consistent

with calculation obtained by using the Abrikosov-fermion mean field theory for a topological insulator[28] and that by the functional-integral saddle-point theory[29].

We also examine the case where $\rho_d|J|$ is large being of order 1 although our method is likely, however, not reliable in this region. In this region, we have a solution for $K(t) \sim 7\zeta(3)/(\pi^2 t^2) + O(1/t^4)$ and T_K is an algebraic function μ such as μ^α with a constant α . If we use $K(t) \simeq 7\zeta(3)/(\pi^2 t^2)$, we obtain

$$\begin{aligned} k_B T_K &= \frac{1}{\pi^2} \sqrt{\frac{7\zeta(3)}{16}} \mu^2 \frac{1}{v} \sqrt{\frac{|J|}{v}} \\ &= \frac{1}{\pi} \sqrt{\frac{7\zeta(3)}{8}} |\mu| \sqrt{\rho_d(\mu)|J|}. \end{aligned} \quad (52)$$

T_K is proportional to $|\mu|$ times an algebraic function of $\rho_d|J|$.

In one dimension ($d = 1$), T_K is obtained in a similar way. For small $\rho_d|J|$, we have

$$k_B T_K = \frac{2e^\gamma D}{\pi} \frac{1}{\sqrt{1 + \frac{D^2}{4\mu^2}}} \exp\left(-8\pi v \frac{1}{|J|}\right). \quad (53)$$

For small μ , T_K is

$$k_B T_K = \frac{4e^\gamma |\mu|}{\pi} \exp\left(-8\pi v \frac{1}{|J|}\right). \quad (54)$$

Let us turn to the two-dimensional case. The integrand is an odd function of k for $d = 2$, and thus the integral is not straightforwardly reduced to a complex integral. The results for $d = 3$ and 1, however, show that k^{d-1} in the integrand is approximately replaced by $(|\mu|/v)^{d-1}$ because the zero of denominators in the integrand gives important contributions. This results in the formula of T_K for $d = 2$.

As a result, the formula of T_K in d dimensions reads as in eq.(50) for $d=1, 2, \dots$. The solution for $\rho_d|J|$ being of order 1 in the region $K(t) \sim 7\zeta(3)/(\pi^2 t^2)$ is given by eq.(52).

We derived T_K by using the Green's function theory. T_K is proportional to $|\mu|$ with the exponential factor for small $\rho_d|J|$ and $|\mu| \ll D$. The formula of T_K shows that T_K decreases as the dimension d is increased. We summarize the results for T_K in Table 1 (the first column).

There has been a calculation based on the numerical renormalization group method[30] for a pseudogap $U = \infty$ Anderson model with the density of states $\rho(\omega) \propto |\omega + \mu|^r$ for $r = 1$ [31]. Their results indicate a particle-hole asymmetry showing $T_K \propto \mu^x$ with $x \simeq 2.6$ for $\mu > 0$ and $T_K \propto |\mu|$ for $\mu < 0$. This kind of asymmetry cannot be visible in our method.

V. ELECTRICAL RESISTIVITY

We consider the conductivity given by the formula:

$$\sigma = -\frac{2e^2}{3} \int \tau_k v_k^2 \frac{\partial f}{\partial \xi_k} \rho_d d\xi_k. \quad (55)$$

TABLE I: Characteristic behaviors of Kondo temperature T_K , the resistivity R and the specific heat ΔC where $\bar{D} = D/\sqrt{1 + D^2/(2\mu)^2}$. We set $K(t) = \text{Re}\psi(1/2 + i\mu/(\pi T)) - \psi(1/2)$ and $I(t) = \text{Im}\psi'(1/2 + i\mu/(\pi T))$, where $\psi(z)$ is the digamma function. α is a constant. We assume that $T \ll D$.

Quantities	Conditions	
T_K	$\rho_d J \ll 1$	$T_K \simeq \bar{D} \exp\left(-\frac{8}{\rho_d(\mu) J }\right)$
	$(\mu \ll D)$	$T_K \simeq \mu \exp\left(-\frac{8}{\rho_d(\mu) J }\right)$
	$\rho_d J \sim O(1)$	$T_K \simeq \mu (\sqrt{\rho_d J } + \dots)$
R	$T_K < T \ll \mu $	$R \simeq \frac{c\pi}{e^2 v^2 \rho_d} J \left(\log\left(\frac{T}{T_K}\right)\right)^{-1}$ $\simeq \frac{c\pi}{e^2 v^2} \frac{1}{8} J ^2 \left(1 + \frac{1}{8} \rho_d J \log \frac{\bar{D}}{T}\right)$
	$T \sim \mu $	$R \simeq \frac{c\pi}{e^2 v^2} J ^2 \left(1 + \frac{1}{8} \rho_d J K(t)\right)$
	$ \mu \ll T$	$R \simeq \frac{c\pi}{e^2 v^2} J ^2$
ΔC	$T_K < T \ll \mu $	$\Delta C \propto (\frac{1}{8} \rho_d J)^{3/2} T \left(\log \frac{T}{T_K}\right)^{-3/2}$ $\propto (\frac{1}{8} \rho_d J)^3 T \left(1 + \frac{3}{16} \rho_d J \log \frac{\bar{D}}{T}\right)$
	$T \sim \mu $	$\Delta C \propto T \frac{\partial^2}{\partial T^2} (C_1 T K(t) + C_2 T I(t))$
	$ \mu \ll T$	$\Delta C \simeq O(\mu/T)$

The life time τ_k is given as

$$\frac{1}{\tau_k} = cN \text{Im} G_{\mathbf{k}\mathbf{k}\uparrow}(\omega + i\delta)^{-1}, \quad (56)$$

where c is the concentration of magnetic impurities. We adopt that $T > T_K$ so that $m_k = 0$. Then we have $\Gamma(\omega) = -(3/4)F(\omega)$ and the Green function is

$$\begin{aligned} G_{\mathbf{k}\mathbf{k}\uparrow}(\omega) &= G_{\mathbf{k}\uparrow}^0(\omega) \left[1 - \left(\frac{J}{2}\right)^2 \frac{3}{4N} F(\omega) G_{\mathbf{k}\uparrow}^0(\omega) \right. \\ &\quad \left. \times \frac{1}{1 + JG(\omega) - \left(\frac{J}{2}\right)^2 \frac{3}{4} F(\omega)^2} \right]. \end{aligned} \quad (57)$$

This is approximately given as

$$G_{\mathbf{k}\mathbf{k}\uparrow}(\omega)^{-1} \simeq G_{\mathbf{k}\uparrow}^0(\omega)^{-1} - \frac{3J^2}{16N} \frac{F(\omega)}{1 + JG(\omega)} + O(J^4). \quad (58)$$

$\text{Im}F(\omega + i\delta)$ is the density of states:

$$\text{Im}F(\omega + i\delta) = -\frac{1}{2} \pi \rho_d(\omega + \mu), \quad (59)$$

where we assume that $\mu > 0$. Then the life time τ_k reads

$$\tau_k(\omega) \simeq \frac{32}{3c\pi J^2 \rho_d(\omega + \mu)} (1 + JG(\omega)). \quad (60)$$

This results in the conductivity:

$$\begin{aligned} \sigma &\simeq \frac{2e^2}{3} v^2 \frac{32}{3c\pi J^2} \left(1 - |J|G(0)\right) \\ &\simeq \frac{2e^2}{3} v^2 \frac{32}{3c\pi J^2} \left(1 + \frac{1}{16} \rho_d(\mu) |J| \right. \\ &\quad \times \left[-2\psi\left(\frac{1}{2} + \frac{D}{2\pi T}\right) + 2\psi\left(\frac{1}{2}\right) \right. \\ &\quad \left. + \psi\left(\frac{1}{2} + \frac{\mu}{\pi iT} + \frac{D}{2\pi T}\right) - \psi\left(\frac{1}{2} + \frac{\mu}{\pi iT}\right) \right. \\ &\quad \left. \left. + \psi\left(\frac{1}{2} - \frac{\mu}{\pi iT} + \frac{D}{2\pi T}\right) - \psi\left(\frac{1}{2} - \frac{\mu}{\pi iT}\right) \right] \right). \end{aligned} \quad (61)$$

We assume $T \ll D$. At low temperatures, when $T \ll |\mu|$, we obtain

$$\sigma \simeq \frac{8}{9c\pi} e^2 v^2 \frac{1}{|J|} \rho_d(\mu) \log\left(\frac{T}{T_K}\right). \quad (62)$$

The resistivity $R = 1/\sigma$ is

$$\begin{aligned} R &= \frac{9}{64} c\pi \frac{1}{e^2 v^2} |J|^2 \left[1 - \frac{\rho_d(\mu)|J|}{8} \log\left(\frac{2e^\gamma \bar{D}}{\pi T}\right) \right]^{-1} \\ &\simeq \frac{9}{64} c\pi \frac{1}{e^2 v^2} |J|^2 \left[1 + \frac{\rho_d(\mu)|J|}{8} \log\left(\frac{2e^\gamma \bar{D}}{\pi T}\right) \right], \end{aligned} \quad (63)$$

with $\bar{D} = D/\sqrt{1 + D^2/(2\mu)^2}$. There is a logarithmic T -dependence in the resistivity, which characterizes the Kondo effect. This is consistent with the result in Ref.[32] up to $\log-T$ term. We must note that the coefficient of the logarithmic term depends on the chemical potential μ .

When $T \sim O(|\mu|)$, R reads

$$\begin{aligned} R &\simeq \frac{9}{64} c\pi \frac{1}{e^2 v^2} |J|^2 \left(1 + \frac{1}{8} \rho_d(\mu) |J| \left[K(t) \right. \right. \\ &\quad \left. \left. + \log\left(\frac{D}{\sqrt{D^2 + 4\mu^2}}\right) \right] \right). \end{aligned} \quad (64)$$

The $\log-T$ dependence is reduced in this region. In the high-temperature region where $|\mu| \ll T$ holds, the $\log(T)$ term will be absent. In the high-temperature region for $|\mu| \ll T$, the resistivity is

$$\begin{aligned} R &\simeq \frac{9}{64} c\pi \frac{1}{e^2 v^2} |J|^2 \left(1 + \frac{1}{8} \rho_d(\mu) |J| \right. \\ &\quad \times \left[7\zeta(3) \frac{1}{\pi^2} \left(\frac{\mu}{T}\right)^2 - 31\zeta(5) \frac{1}{\pi^4} \left(\frac{\mu}{T}\right)^4 + \dots \right] \right). \end{aligned} \quad (65)$$

This has no $\log-T$ dependence. As a result $\log-T$ appears in the region $T_K < T \ll |\mu|$. We summarize the results for R in the second column of Table 1. The $\log-T$ dependence is reduced as the chemical potential $|\mu|$ approaches the Dirac point, that is, $\log-T$ will disappear as $\mu \rightarrow 0$.

VI. SPECIFIC HEAT

Let us examine the specific heat in this section. The specific heat also shows a singularity as in other physical quantities. We calculate the additional entropy coming from the s-d interaction with magnetic impurities. The expectation value of the interaction part H_{sd} is given by

$$\begin{aligned} V &\equiv \langle H_{sd} \rangle \\ &= \frac{J}{2N\beta} \sum_{\mathbf{k}\mathbf{k}'n\sigma} \Gamma_{\mathbf{k}\mathbf{k}'\sigma}(i\omega_n) \\ &= \frac{J}{N\beta} \sum_{\mathbf{k}n\sigma} G_{\mathbf{k}\sigma}^0(i\omega_n) t(i\omega_n) = \frac{2J}{N\beta} \sum_{\mathbf{k}n} G_{\mathbf{k}}^0(i\omega_n) t(i\omega_n), \end{aligned} \quad (66)$$

where $t(z)$ is defined as

$$t(z) = -\frac{J}{4} \frac{\Gamma(z)}{1 + JG(z) + (J/2)^2 \Gamma(z)F(z)}. \quad (67)$$

The ω_n -summation is performed to give

$$\begin{aligned} V &= 2J \frac{1}{N} \sum_{\mathbf{k}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega f(\omega) \left[\right. \\ &\quad \text{Re} \left(\frac{1}{\omega + \mu + vk} + \frac{1}{\omega + \mu - vk} \right) \text{Im} t(\omega - i\delta) \\ &\quad \left. + i\pi \left(\delta(\omega + \mu + vk) + \delta(\omega + \mu - vk) \right) \text{Re} t(\omega - i\delta) \right]. \end{aligned} \quad (68)$$

Because the ω -dependence in $t(\omega)$ is important, we neglect the ω -dependence in $1/(\omega + \mu \pm vk)$ and $\delta(\omega + \mu \pm vk)$. We adopt $m_k = 0$ ($T > T_K$), so that $\Gamma(\omega - i\delta) = -(3/4)F(\omega - i\delta)$. $F(\omega - i\delta)$ is approximated as $F(\omega - i\delta) \simeq F(-i\delta) \simeq \rho_d(\mu)F_d + i\pi\rho_d(\mu)/2$, where $F_d = -\text{sign}(\mu)v\Lambda/|\mu|$ for $d = 3$ and $F_d = -\text{sign}(\mu)(1/2)\ln|(v^2\Lambda^2 - \mu^2)/\mu^2|$ for $d = 2$ with a cutoff Λ in k -integration. The real part of $F(\omega - i\delta)$ is proportional to $\mu/|\mu|$. Then the interaction energy is

$$V = \frac{3}{8} F_d \left(\rho_d(\mu) J \right)^2 \int_{-\infty}^{\infty} d\omega f(\omega) \frac{1}{1 + JG(\omega)}. \quad (69)$$

Using the relation between the free energy and V given as

$$V = J \frac{\partial F}{\partial J}, \quad (70)$$

the additional entropy is obtained as

$$\Delta S = -\frac{\partial}{\partial T}(F - F_0) = -\int_0^J \frac{dJ'}{J'} \frac{\partial}{\partial T} V(J', T). \quad (71)$$

We employ the expansion formula for the Fermi distribution function $f(\omega)$:

$$\int_D^D d\omega f(\omega) h(\omega) = \int_D^0 d\omega h(\omega) + \frac{\pi^2}{6} (k_B T)^2 h'(0), \quad (72)$$

for a differentiable function $h(\omega)$. Then we have

$$V \simeq \frac{3}{8} F_d (\rho_d(\mu) J)^2 \left[\int_{-D}^0 d\omega \frac{1}{1 + JG(\omega)} + \frac{\pi^2}{6} (k_B T)^2 \frac{\partial}{\partial \omega} \frac{1}{1 + JG(\omega)} \Big|_{\omega=0} \right]. \quad (73)$$

We first examine the low-temperature region characterized by $T \ll |\mu|$. When $|\mu|/T$ is large, the second term of V is written as

$$V_2 = -\frac{\pi^2}{2} F_d \rho_d(\mu) |J| (k_B T)^2 \frac{1}{\left(\ln(T_K/T) - g(\omega) \right)^2} g'(\omega) \Big|_{\omega=0}, \quad (74)$$

where $g(\omega)$ is defined as

$$g(\omega) = \ln \left(\frac{T_K}{T} \right) + \frac{8}{\rho_d(\mu) |J|} (1 + JG(\omega)). \quad (75)$$

We have $g(0) = 0$ and $g'(0)$ is evaluated as

$$\begin{aligned} g'(0) &\simeq -\frac{1}{2} \left[\frac{1}{2\pi i T} \psi' \left(\frac{1}{2} + \frac{\mu}{\pi i T} \right) - \frac{1}{2\pi i T} \psi' \left(\frac{1}{2} - \frac{\mu}{\pi i T} \right) \right. \\ &\quad + \frac{1}{2\pi i T} \psi' \left(\frac{1}{2} - \frac{\mu}{\pi i T} + \frac{D}{2\pi T} \right) \\ &\quad \left. - \frac{1}{2\pi i T} \psi' \left(\frac{1}{2} + \frac{\mu}{\pi i T} + \frac{D}{2\pi T} \right) \right] \\ &\simeq -\frac{1}{2\mu} \frac{D^2}{D^2 + 4\mu^2}. \end{aligned} \quad (76)$$

This is in contrast to the case of conventional Kondo effect where $g'(0)$ is proportional to the inverse of temperature $1/T$. Then we obtain

$$V_2 = \frac{\pi^2}{4} A_D \rho_d(\mu) |J| \frac{F_d}{\mu} (k_B T)^2 \frac{1}{\left(\ln(T_K/T) \right)^2}, \quad (77)$$

with A_D is a constant given by $D^2/(D^2 + 4\mu^2)$. The first term of V has a similar singularity[22]. We expand this term in terms of the inverse of $\ln(T_K/T)$ to obtain

$$V_1 = -3F_d \rho_d(\mu) |J| \frac{1}{\left(\ln(T_K/T) \right)^2} \int_{-D}^0 d\omega g(\omega) + \dots \quad (78)$$

$g(\omega)$ is expanded by means of ω/T by restricting the integral region to $(-k_B T, 0)$ to give a term $(k_B T)^2/(\ln(T_K/T))^2$. This contribution is included in the coefficient A_D such as $A_D = (1 - 3/\pi^2) D^2/(D^2 + 4\mu^2)$ and we have

$$V = \frac{\pi^2}{4} A_D \rho_d(\mu) |J| \frac{F_d}{\mu} (k_B T)^2 \frac{1}{\left(\ln(T_K/T) \right)^2}. \quad (79)$$

Because of the relation $\ln T_K = \ln(2e^\gamma \bar{D}/\pi) + 8/(\rho_d J)$, the following holds: $d \ln T_K = -8d(\rho_d J)/(\rho_d J)^2$. Using this, the contribution to the free energy is evaluated as

$$\begin{aligned} \Delta F &= \int_0^{\rho_d J} \frac{d(\rho_d J)}{\rho_d J} V(\rho_d J) \\ &= -2\pi^2 A_D \frac{F_d}{\mu} (k_B T)^2 \frac{1}{\left(\ln(\bar{D}/T) \right)^2} \left[\frac{\rho_d J}{8} \right. \\ &\quad \left. + \frac{1}{\ln(T_K/T)} - \frac{2}{\ln(\bar{D}/T)} \ln \left| \frac{\rho_d J}{8} \ln \left(\frac{T_K}{T} \right) \right| \right], \end{aligned} \quad (80)$$

where we neglected the factor $2e^\gamma/\pi$ for simplicity. The entropy $\Delta S = -\partial \Delta F / \partial T$ is found to be

$$\begin{aligned} \Delta S &= 4\pi^2 A_D k_B F_d \frac{k_B T}{\mu} \left[\frac{1}{3} \left(\frac{\rho_d J}{8} \right)^3 - \frac{1}{2} \left(\frac{\rho_d J}{8} \right)^4 \ln \frac{\bar{D}}{T} \right. \\ &\quad \left. - \frac{3}{5} \left(\frac{\rho_d J}{8} \right)^5 \ln \frac{\bar{D}}{T} + \frac{3}{5} \left(\frac{\rho_d J}{8} \right)^5 \left(\ln \frac{\bar{D}}{T} \right)^2 + \dots \right]. \end{aligned} \quad (81)$$

The entropy has a singularity such as $T \ln T$ in the fourth order of $\rho_d J$, which is rather weak singularity compared to the conventional Kondo effect with $\ln T$ term. This results in the specific heat $\Delta C = T \partial \Delta S / \partial T$:

$$\begin{aligned} \frac{\Delta C}{k_B} &\simeq \frac{4\pi^2}{3} A_D F_d \frac{k_B T}{\mu} \left(\frac{\rho_d J}{8} \right)^3 \left[1 - \frac{3}{2} \left(\frac{\rho_d J}{8} \right) \ln \frac{\bar{D}}{T} + \dots \right] \\ &\simeq \frac{4\pi^2}{3} A_D |F_d| \frac{k_B T}{|\mu|} \left(\frac{\rho_d |J|}{8} \right)^{3/2} \frac{1}{\left(\ln(T/T_K) \right)^{3/2}}. \end{aligned} \quad (82)$$

Hence, the specific heat in a Dirac system has a singularity as a function of the temperature near T_K . This should be compared with the specific heat anomaly in the original Kondo effect given as $\Delta C/k_B \simeq 1/(\ln(T_K/T))^4$, which is shown by following the above method for the original s-d model. The singularity in a Dirac system is much weaker than that of the original Kondo effect. The log- T appears in the range $T_K < T \ll |\mu|/k_B$.

Let us then examine the specific heat in the intermediate region $T \sim |\mu|$. The expectation value of the interaction term V in eq.(73) is evaluated by expanding $JG(\omega)$

in ω and the integral is restricted to the interval $(-T, 0)$. $JG(\omega)$ is written as

$$JG(\omega) = JG(0) + \frac{1}{16}\rho_d|J|\frac{\omega}{T}\text{Im}\psi'\left(\frac{1}{2} + i\frac{\mu}{\pi T}\right), \quad (83)$$

and the main contributions to V are

$$V \simeq F_d(\rho_d|J|)^3 \left[C_1 TK(t) + C_2 T \text{Im}\psi'\left(\frac{1}{2} + i\frac{\mu}{\pi T}\right) \right], \quad (84)$$

where C_1 and C_2 are constants. This results in the specific heat given as

$$\frac{\Delta C}{k_B} \simeq -\frac{1}{3}F_d(\rho_d(\mu)|J|)^3 k_B T \frac{\partial^2}{\partial T^2} (C_1 TK(t) + C_2 TI(t)), \quad (85)$$

where we defined

$$I(t) = \text{Im}\psi'\left(\frac{1}{2} + i\frac{\mu}{\pi T}\right). \quad (86)$$

Thus the log- T terms do not show up in the region $T \sim |\mu|$.

In the high-temperature region defined by $|\mu| \ll T$, the temperature dependence of V comes from the terms of order μ/T . Thus the additional entropy is also of order μ/T and is negligible in the region $|\mu| \ll T$.

VII. SUMMARY

We investigated the Kondo problem with the localized spin which couples to Dirac fermions. The Kondo temperature T_K was calculated from a singularity of Green's functions. The logarithmic terms in the resistivity and the specific heat were derived in the low-temperature region, where the region $k_B T \ll |\mu|$ is called the low-temperature region. We considered two regions:

(1) When $k_B T \ll |\mu|$, the Kondo screening occurs

with the characteristic temperature scale

$$k_B T_K \simeq \bar{D} \exp\left(-\frac{8}{\rho_d(\mu)|J|}\right), \quad (87)$$

for small $\rho_d(\mu)|J| \ll 1$. This is the conventional form, as in the original Kondo problem, being proportional to $\exp(-\text{const}/\rho|J|)$ with the density of states ρ . When $|\mu|$ is small compared to the cutoff D , T_K is proportional to $|\mu|$:

$$k_B T_K \simeq |\mu| \exp\left(-\frac{8}{\rho_d(\mu)|J|}\right). \quad (88)$$

In the range $T_K < T \ll |\mu|/k_B$, the log- T anomaly appears in the physical quantities. For large $\rho_d|J|$ of order 1, T_K will be given by an algebraic function $Q(x)$ such as

$$k_B T_K \simeq |\mu| Q(\rho_d(\mu)|J|). \quad (89)$$

(2) When $k_B T$ is not much smaller than $|\mu|$, namely, $k_B T \sim |\mu|$ or $k_B T > |\mu|$, the Kondo effect is suppressed and the resistivity and specific heat do not exhibit a logarithmic (log- T) anomaly. T_K vanishes when the chemical potential is just at the Dirac point, leading to the absence of Kondo screening.

The vanishing of T_K when μ is at the Dirac point is consistent with the results for the pseudogap Kondo problem[18], and suggests the existence of a phase transition, as $|J|$ is increased, from non-screening phase to screening phase. The term being proportional to σ_z like the magnetic field would also be important in the Kondo effect, which we did not consider in this paper. It is also interesting to study the nature of interaction between two magnetic impurities in Dirac metals. This issue was investigated intensively in the original Kondo problem[33–36]. These are future subjects.

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